MODEL ANSWERS

Please check the examination details below before entering your candidate information

Candidate surname

Other names

**Pearson Edexcel
Level 3 GCE**

Centre Number

Candidate Number

Monday 3 June 2019

Morning (Time: 1 hour 30 minutes)

Paper Reference **9FM0/01**

Further Mathematics

**Advanced
Paper 1: Core Pure Mathematics 1**

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided
  – *there may be more space than you need.*
- You should show sufficient working to make your methods clear.
  Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 8 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets
  – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over ▶

P 6 1 1 7 7 A 0 1 3 2

Pearson

Answer ALL questions. Write your answers in the spaces provided.
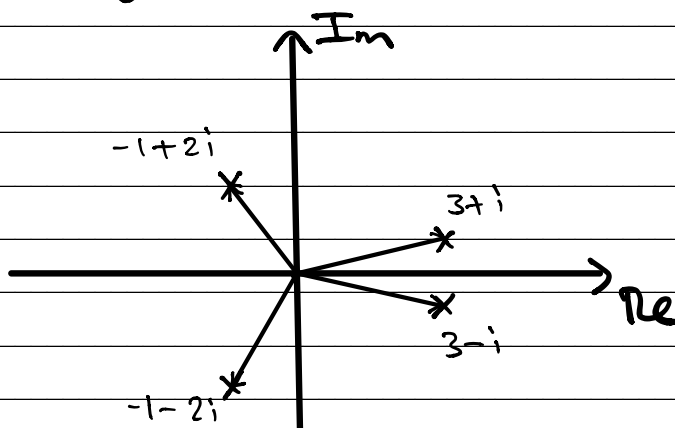
1.
$$f(z) = z^4 + az^3 + bz^2 + cz + d$$

where a, b, c and d are real constants.

Given that $-1 + 2i$ and $3 - i$ are two roots of the equation $f(z) = 0$

(a) show all the roots of $f(z) = 0$ on a single Argand diagram,

(4)

(b) find the values of a, b, c and d.

(5)

a) $-1-2i$ and $3+i$ are also roots
(conjugate pairs)

b) $f(z) =$
$(z-(-1+2i))(z-(-1-2i))(z-(3+i))(z-(3-i))$
$= (z+1-2i)(z+1+2i)(z-3-i)(z-3+i)$
$= (z^2+2z+5)(z^2-6z+10)$

$= z^4-6z^3+10z^2+2z^3-12z^2+20z+5z^2$
$\quad -30z+50$

$= z^4-4z^3+3z^2-10z+50 = f(z)$

So $\boxed{a=-4, \ b=3, \ c=-10}$

2. Show that

$$I = \int_0^\infty \frac{8x-12}{(2x^2+3)(x+1)} \, dx = k$$

where k is a rational number to be found.

(7)

Partial Fractions :

$$\frac{8x-12}{(2x^2+3)(x+1)} \equiv \frac{Ax+B}{2x^2+3} + \frac{C}{x+1}$$

$$8x-12 \equiv (Ax+B)(x+1) + C(2x^2+3)$$

let $x = -1$: $\quad -20 = C(5)$

$$\therefore C = -4$$

let $x = 0$: $\quad -12 = B(1) - 4(3)$

$$\therefore B = 0$$

let $x = 1$: $\quad -4 = A(2) - 4(5)$

$$\therefore A = 8$$

$$\therefore I = \int_0^\infty \frac{8x}{2x^2+3} - \frac{4}{x+1} \, dx$$

use reverse chain rule

$$= \lim_{c \to \infty} \left[2\ln|2x^2+3| - 4\ln|x+1| \right]_0^c$$

$$= \lim_{c \to \infty} \left[\ln(2x^2+3)^2 - \ln(x+1)^4 \right]_0^c$$

Question 2 continued

$$= \lim_{c \to \infty} \left[\ln \left[\frac{(2x^2+3)^2}{(x+1)^4} \right] \right]_0^c$$

$$= \lim_{c \to \infty} \left[\ln \left[\frac{(2c^2+3)^2}{(c+1)^4} \right] \right] - \ln 9$$

$$= \quad \ln 4 - \ln 9 = \boxed{\ln \frac{4}{9}}$$

3.

$\theta = \pi/2$

A — 1.2m — B

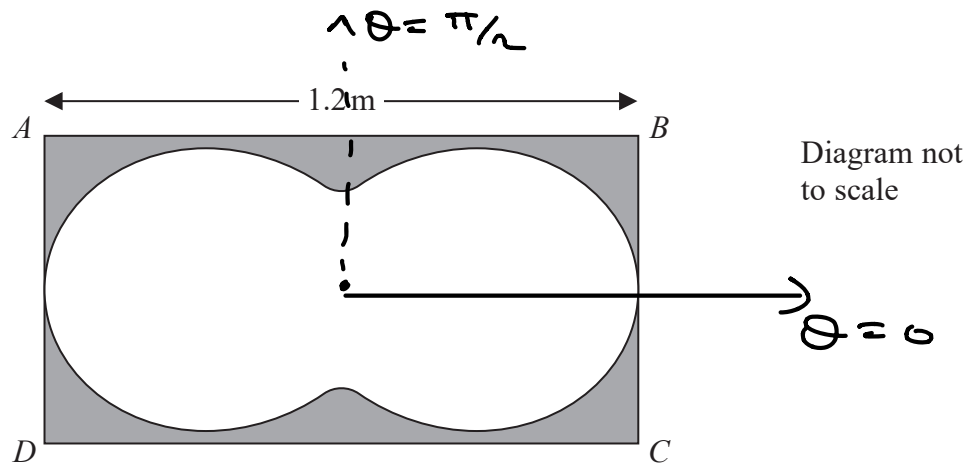Diagram not to scale

$\theta = 0$

D C

Figure 1

Figure 1 shows the design for a table top in the shape of a rectangle $ABCD$. The length of the table, AB, is 1.2 m. The area inside the closed curve is made of glass and the surrounding area, shown shaded in Figure 1, is made of wood.

The perimeter of the glass is modelled by the curve with polar equation

$$r = 0.4 + a\cos 2\theta \qquad 0 \leqslant \theta < 2\pi$$

where a is a constant.

(a) Show that $a = 0.2$

(2)

Hence, given that $AD = 60$ cm,

(b) find the area of the wooden part of the table top, giving your answer in m^2 to 3 significant figures.

(8)

a) $\theta = 0, \ r = 0.6 \ : \ 0.6 = 0.4 + a$

$\therefore a = 0.2$

b) Area total $= 0.6 \times 1.2 = 0.72 \, m^2$

Area glass $= 2 \cdot \frac{1}{2} \int_0^{\pi} (0.4 + 0.2\cos 2\theta)^2 \, d\theta$

$= \int_0^{\pi} 0.16 + 0.04\cos^2 2\theta + 0.16\cos 2\theta \, d\theta$

Question 3 continued

$$\cos 4\theta = 2\cos^2(2\theta) - 1$$

$$\frac{\cos 4\theta + 1}{2} = \cos^2 2\theta$$

$$\Rightarrow \text{Area}_{glass} = \int_0^\pi 0.16 + \frac{0.04}{2}(\cos 4\theta + 1) + 0.16\cos 2\theta \, d\theta$$

$$\Rightarrow \text{Area}_{glass} = \int_0^\pi 0.18 + 0.16\cos 2\theta + 0.02\cos 4\theta \, d\theta$$

$$= \left[0.18\theta + 0.08\sin 2\theta + 0.005\sin 4\theta \right]_0^\pi$$

$$= \left[0.18\pi + 0 + 0 \right] - \left[0 \right]$$

$$= 0.18\pi \; //$$

$$\therefore \text{Shaded region} = 0.72 - 0.18\pi$$

$$= \boxed{0.155} \; m^2$$

4. Prove that, for $n \in \mathbb{Z}, n \geqslant 0$

$$\sum_{r=0}^{n} \frac{1}{(r+1)(r+2)(r+3)} = \frac{an^2 + bn}{c(n+2)(n+3)}$$

where a, b and c are integers to be found.

(5)

$$1 = A(r+2)(r+3) + B(r+1)(r+3) + C(r+1)(r+2)$$

$r = -1 : \quad 1 = A(1)(2) \qquad \therefore A = \frac{1}{2}$

$r = -2 : \quad 1 = B(-1)(1) \qquad \therefore B = -1$

$r = 0 : \quad 1 = 2C$

$\qquad \therefore C = \frac{1}{2}$

$\Rightarrow \quad \dfrac{\frac{1}{2}}{r+1} - \dfrac{1}{r+2} + \dfrac{\frac{1}{2}}{r+3}$

$\Rightarrow \quad \dfrac{1}{2}\left[\dfrac{1}{r+1} - \dfrac{2}{r+2} + \dfrac{1}{r+3} \right]$

$f(0) \quad \dfrac{1}{1} - \dfrac{2}{2} + \dfrac{1}{3}$

$f(1) \quad \boxed{\dfrac{1}{2}} - \dfrac{2}{3} + \dfrac{1}{4}$

$f(2) \quad \dfrac{1}{3} - \dfrac{2}{4} + \dfrac{1}{5}$

Question 4 continued

$$f(3) \quad \frac{1}{4} - \frac{2}{5} + \frac{1}{6}$$

$$\vdots$$

$$f(n-1) \quad \frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1}$$

$$f(n) \quad \frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}$$

$$f(n+1) \quad \frac{1}{n+1} - \frac{2}{n+2} + \frac{1}{n+3}$$

$$\Rightarrow \quad \frac{1}{2}\left[\frac{1}{2} - \frac{1}{n+2} + \frac{1}{n+3} \right]$$

$$\Rightarrow \quad \frac{1}{2}\left[\frac{(n+2)(n+3) - 2(n+3) + 2(n+2)}{2(n+2)(n+3)} \right]$$

$$\Rightarrow \quad \frac{n^2 + 5n + 6 - 2n - 6 + 2n + 4}{4(n+2)(n+3)}$$

$$\Rightarrow \quad \frac{n^2 + 5n + 4}{4(n+2)(n+3)} = \boxed{\frac{(n+4)(n+1)}{4(n+2)(n+3)}}$$

5. A tank at a chemical plant has a capacity of 250 litres. The tank initially contains 100 litres of pure water.

Salt water enters the tank at a rate of 3 litres every minute. Each litre of salt water entering the tank contains 1 gram of salt.

It is assumed that the salt water mixes instantly with the contents of the tank upon entry.

At the instant when the salt water begins to enter the tank, a valve is opened at the bottom of the tank and the solution in the tank flows out at a rate of 2 litres per minute.

Given that there are S grams of salt in the tank after t minutes,

(a) show that the situation can be modelled by the differential equation

$$\frac{dS}{dt} = 3 - \frac{2S}{100 + t}$$

(4)

(b) Hence find the number of grams of salt in the tank after 10 minutes.

(5)

When the concentration of salt in the tank reaches 0.9 grams per litre, the valve at the bottom of the tank must be closed.

(c) Find, to the nearest minute, when the valve would need to be closed.

(3)

(d) Evaluate the model.

(1)

a) $(mass)_{in} = 3$ $(per\ minute)$

$(mass)_{out} = 2 \times \left(\dfrac{volume\ of\ salt\ water\ added}{total\ volume} \right)$

because 2 litres flows out per min.

every min, 2l flows out but 3l comes in ∴ there is a net 1l added every min. So amount of water = 100+t (at time t)

given by S

$\therefore (mass)_{out} = 2 \times \left(\dfrac{S}{100 + t} \right)$

Question 5 continued

$$\frac{dS}{dt} = (mass)_{in} - (mass)_{out}$$

$$\Rightarrow \frac{dS}{dt} = 3 - \frac{2S}{100+t}$$

b) $\quad \frac{dS}{dt} + S\left(\frac{2}{100+t}\right) = 3$

$$I = e^{\int \frac{2}{100+t} dt} = e^{2\ln|100+t|} = e^{\ln(100+t)^2}$$

$$= (100+t)^2$$

$$\times (100+t)^2 : \quad (100+t)^2 \frac{dS}{dt} + 2S(100+t) = 3(100+t)^2$$

$$\frac{d}{dt}\left(S(100+t)^2\right) = 3(100+t)^2$$

$$S(100+t)^2 = 3\int (100+t)^2 \, dt$$

$$S(100+t)^2 = 3\left[\frac{(100+t)^3}{3}\right] + C$$

Question 5 continued

$$\therefore \quad S(100+t)^2 = (100+t)^3 + c$$

$t=0, S=0 : \qquad 0 = (100)^3 + c$

$$\therefore \quad c = -(100)^3$$

$$\Rightarrow \quad S(100+t)^2 = (100+t)^3 - (100)^3$$

$$\Rightarrow \quad S = (100+t) - \frac{100^3}{(100+t)^2}$$

$t=10 : \qquad S = (110) - \frac{100^3}{110^2}$

$$= \boxed{27.4 g}$$

c) $\Rightarrow \quad \dfrac{S}{100+t} = 0.9$

$$S = 90 + 0.9t$$

from (b), $\quad S = 100+t - \dfrac{100^3}{(100+t)^2}$

$$\Rightarrow \quad 100+t - \frac{100^3}{(100+t)^2} = 90 + 0.9t$$

$$\Rightarrow \quad 10 + 0.1t = \frac{100^3}{(100+t)^2}$$

$$\Rightarrow \quad \frac{1}{10}(100+t) = \frac{100^3}{(100+t)^2}$$

$$\Rightarrow \quad \frac{1}{10}(100+t)^3 = 100^3$$

$$(100+t)^3 = 10(100^3)$$

$$100+t = \sqrt[3]{10(100^3)}$$

$$t = \sqrt[3]{10(100)^3} - 100$$

$$= 100 \sqrt[3]{10} - 100$$

$$= 100\left(\sqrt[3]{10} - 1\right)$$

$$= 115.4 \;//$$

so $\boxed{115}$ minutes.

d) The model is fine for $0 < t < 115$ but it should not be valid for $t > 115$, as the valve closes at $t = 115$!

$\Bigg($ at $t = 115$, volume of solution in tank is 194 litres, which is less than the capacity of 250 litres. So we don't need to worry about the capacity. $\Bigg)$

(Total for Question 5 is 13 marks)

6. Prove by induction that for all positive integers n

$$f(n) = 3^{2n+4} - 2^{2n}$$

is divisible by 5

(6)

$$\underline{n=1}: \quad f(1) = 3^6 - 2^4 = 725 = 145(5)$$
$$\therefore \text{true for } n=1 \quad /\!/$$

$\underline{\text{assume true for } n=u}:$

$$\text{ie } f(u) = 3^{2u+4} - 2^{2u} \text{ is divisible by 5}$$

$\underline{\text{consider } n=u+1}:$

$$f(u+1) = 3^{2(u+1)+4} - 2^{2(u+1)}$$
$$= 3^{2u+6} - 2^{2u+2}$$
$$= 3^2(3^{2u+4}) - 2^2(2^{2u})$$
$$= 9(3^{2u+4}) - 4(2^{2u})$$
$$= 9\left[3^{2u+4} - 2^{2u}\right] + 5(2^u)$$
$$= 9f(u) + 5(2^u)$$

$\uparrow$ div by 5 (assumption) $\quad \curvearrowright$ div by 5

$$\therefore \text{true for } n=u+1 \quad =$$

So: · we proved the statement true for $n=1$.
   · when assumed true for $n=u$, we proved
   it to be true for $n=u+1$.
   · $\therefore$ by mathematical induction it's true for all
   $n \in \mathbb{Z}^+$.

P 6 1 1 7 7 A 0 2 2 3 2

7. The line l_1 has equation

$$\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-4}{3}$$

The line l_2 has equation

$$\mathbf{r} = \mathbf{i} + 3\mathbf{k} + t(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$

where t is a scalar parameter.

(a) Show that l_1 and l_2 lie in the same plane.

(3)

(b) Write down a vector equation for the plane containing l_1 and l_2

(1)

(c) Find, to the nearest degree, the acute angle between l_1 and l_2

(3)

a) Two lines are coplanar if they meet, or are parallel. l_1 and l_2 are clearly not parallel since their direction vectors aren't multiples of one another, so lets show that they meet:

$$l_1 \rightarrow \begin{pmatrix} 1 + 2t \\ -1 - t \\ 4 + 3t \end{pmatrix} = \begin{pmatrix} 1 + s \\ -s \\ 3 + 2s \end{pmatrix} \leftarrow l_2$$

equating i: $\quad 1 + 2t = 1 + s$
$$\therefore s = 2t$$

equating j: $\quad -1 - t = -s$
$$\text{but } s = 2t$$
$$\therefore -1 - t = -2t$$
$$t = 1, \quad \text{so } s = 2$$

equating k: $\quad 4 + 3t = 3 + 2s$

$$\text{using } t = 1: \quad 4 + 3 = 3 + 2s$$
$$4 = 2s \therefore s = 2 \checkmark$$

Question 7 continued

Values of t & s are consistent so l_1 & l_2 do indeed meet. Hence they are coplanar.

b) $r = a + \lambda b + \mu c$

where b, c are vectors that lie on the plane, and a is a point on the plane.

$$a = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad c = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$\Rightarrow \boxed{r = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}}$$

c) $\cos\theta = \dfrac{a \cdot b}{|a||b|}, \quad a = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ & $b = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

direction vectors

$$a \cdot b = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 9$$

$$|a| = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$$

$$|b| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$$

Question 7 continued

$$\Rightarrow \quad \cos\theta = \frac{9}{\sqrt{14} \cdot \sqrt{6}} \qquad \therefore \theta = \arccos\left(\frac{9}{\sqrt{14} \cdot \sqrt{6}}\right)$$

$$= 10.9°$$

$$= \boxed{11°}$$

8. A scientist is studying the effect of introducing a population of white-clawed crayfish into a population of signal crayfish.
At time t years, the number of white-clawed crayfish, w, and the number of signal crayfish, s, are modelled by the differential equations

$$\frac{dw}{dt} = \frac{5}{2}(w - s)$$
$$\frac{ds}{dt} = \frac{2}{5}w - 90e^{-t}$$

(a) Show that

$$2\frac{d^2w}{dt^2} - 5\frac{dw}{dt} + 2w = 450e^{-t}$$

(3)

(b) Find a general solution for the number of white-clawed crayfish at time t years.

(6)

(c) Find a general solution for the number of signal crayfish at time t years.

(2)

The model predicts that, at time T years, the population of white-clawed crayfish will have died out.

Given that $w = 65$ and $s = 85$ when $t = 0$

(d) find the value of T, giving your answer to 3 decimal places.

(6)

(e) Suggest a limitation of the model.

(1)

a) $\quad \dfrac{dw}{dt} = \dfrac{5}{2}w - \dfrac{5}{2}s$

$\quad w - \dfrac{2}{5}\dfrac{dw}{dt} = s$

$\quad \dfrac{ds}{dt} = \dfrac{dw}{dt} - \dfrac{2}{5}\dfrac{d^2w}{dt^2}$

but $\quad \dfrac{ds}{dt} = \dfrac{2}{5}w - 90e^{-t}$

$$\Rightarrow \frac{2}{5}w - 90e^{-t} = \frac{dw}{dt} - \frac{2}{5}\frac{d^2w}{dt^2}$$

$$\times \frac{5}{2}: \quad w - 225e^{-t} = \frac{5}{2}\frac{dw}{dt} - \frac{d^2w}{dt^2}$$

$$\Rightarrow \frac{d^2w}{dt^2} - \frac{5}{2}\frac{dw}{dt} + w = 225e^{-t}$$

$$\times 2: \quad 2\frac{d^2w}{dt^2} - 5\frac{dw}{dt} + 2w = 450e^{-t} \quad \text{//}$$

b) $\underline{AUX}: \quad 2\lambda^2 - 5\lambda + 2 = 0$

$$(2\lambda - 1)(\lambda - 2) = 0$$

$$\lambda = \frac{1}{2}, \qquad \lambda = 2 \quad \text{//}$$

$\therefore \underline{C.F}: \quad w = Ae^{\frac{1}{2}t} + Be^{2t}$

$\underline{P.I}: \quad$ let $w = Ce^{-t}$

$$\dot{w} = -Ce^{-t}$$
$$\ddot{w} = Ce^{-t}$$

$\underline{\text{subbing the P.I back into the DE}:}$

$$2Ce^{-t} + 5Ce^{-t} + 2Ce^{-t} = 450e^{-t}$$

Question 8 continued

<u>compare coefficients of e^{-t}:</u>

$$2C + 5C + 2C = 450$$
$$9C = 450 \quad \therefore \quad C = 50$$

so $\quad \omega = 50e^{-t}$

G.S : $\boxed{\omega = Ae^{\frac{1}{2}t} + Be^{2t} + 50e^{-t}}$

c) <u>from a</u> : $\quad S = \omega - \frac{2}{5}\frac{d\omega}{dt}$

$$\frac{d\omega}{dt} = \frac{1}{2}Ae^{\frac{1}{2}t} + 2Be^{2t} - 50e^{-t}$$

$\therefore$ $S = Ae^{\frac{1}{2}t} + Be^{2t} + 50e^{-t} - \frac{2}{5}\left(\frac{1}{2}Ae^{\frac{1}{2}t} + 2Be^{2t} - 50e^{-t}\right)$

$\Rightarrow \quad S = Ae^{\frac{1}{2}t} - \frac{1}{5}Ae^{\frac{1}{2}t} + Be^{2t} - \frac{4}{5}Be^{2t}$
$\qquad\qquad + 50e^{-t} + 20e^{-t}$

$\Rightarrow \boxed{S = \frac{4}{5}Ae^{\frac{1}{2}t} + \frac{1}{5}Be^{2t} + 70e^{-t}}$

d) $\omega = 65, t = 0$: $\quad 65 = A + B + 50$

$\qquad\qquad\qquad \therefore A + B = 15 \quad — \textcircled{1}$

$S = 85, t = 0$: $\quad 85 = \frac{4A}{5} + \frac{B}{5} + 70$

$\qquad\qquad\qquad \therefore 4A + B = 75 \quad — \textcircled{2}$

Question 8 continued

② − ① : $3A = 60$

$$\therefore A = 20$$

and from ① , $B = 15 - A$
$$= 15 - 20$$
$$= -5$$

So $w = 20e^{\frac{1}{2}t} - 5e^{2t} + 50e^{-t}$

$w = 0, t = T$ ∴ $0 = 20e^{\frac{1}{2}T} - 5e^{2T} + 50e^{-T}$

$$e^{-T}\left[50 - 5e^{3T} + 20e^{\frac{3T}{2}}\right] = 0$$

$e^{-T} \neq 0$ so $50 - 5e^{3T} + 20e^{\frac{3T}{2}} = 0$

This is a quadratic in $e^{\frac{3T}{2}}$,

let $y = e^{3T/2}$.

Then $50 - 5y^2 + 20y = 0$

$$5y^2 - 20y - 50 = 0$$

$$y^2 - 4y - 10 = 0$$

$$y = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(-10)}}{2}$$

$e^{3T/2} = \frac{4 \pm \sqrt{56}}{2}$ $\Bigg|$ $\ln e^{3T/2} = \ln\left(\frac{4 \pm \sqrt{56}}{2}\right)$

$$3T/2 = \ln\left(\frac{4 \pm \sqrt{56}}{2}\right)$$

Question 8 continued $T = \dfrac{2}{3} \ln \left(\dfrac{4 \pm \sqrt{56}}{2} \right) = 1.165$ ✳

e) for $t > \dfrac{2}{3} \ln S$, ω will be negative which does not make any sense since ω is a population. S will also be negative for large enough t.

<div align="center">OR</div>

S will likely grow after ω die out due to less competition for resources. this is not supported by the model.

(Total for Question 8 is 18 marks)

TOTAL FOR PAPER IS 75 MARKS